On Some Loci Associated with Plane Curves.

By Dr. C. H. Sisam.

1. Let there be given two algebraic curves C and C_1, lying in the same plane. Let t (Fig. 1) be a tangent to C_1. At the points of intersection of t and C construct the tangents to C. Then as t describes the system of tangents to C_1, the points of intersection of the tangents to C describe a locus, some of the properties of which will now be determined. This locus will be referred to as the locus C_1'. The given curves C and C_1 will be supposed to have only the ordinary point and line singularities and to lie in general position with respect to each other.

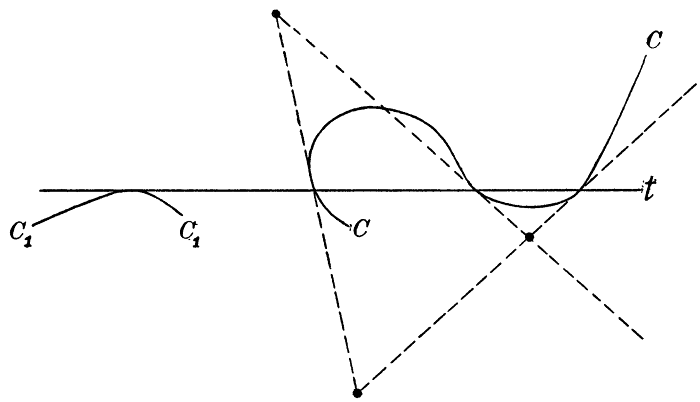

Fig. 1.

2. To determine the order of C_1', consider first the particular case in which C_1' reduces to the pencil of lines through a point P. The locus so obtained is a direct generalization, to curves of order higher than the second, of the polar of a point with respect to a conic, according to a frequent definition of that locus. Let it be referred to as the locus C_P'.

3. The equation of C_P involves not only the current coordinates (x, y, z) of a point on the locus, but also the coordinates (x', y', z') of P. Its order in

(x', y', z') is* $\frac{1}{2}n(n-1)$, since it is the equation of the cords of contact of the tangents from (x, y, z) to C. To determine its order in (x, y, z), let $x' = \rho x$, $y' = \rho y$, $z' = \rho z$. We then obtain the equation of the locus of the points each of which lies upon its own C'_P locus. To this locus belongs each double and inflectional tangent of C. Every other point of the locus must lie upon C. But C is an $(n-1)$-fold component of the locus, for, if P lies upon C, the tangent at P is an $(n-1)$-fold component of C_P. Hence

$$\tfrac{1}{2}n(n-1) + m'_P = \tau + \iota + m(n-1).$$

Solving for m'_P and simplifying by means of Plucker's equations, we obtain:†

$$m'_P = (m-1)(n-1) - p.$$

This expression is unchanged by duality. If, in fact, we denote by C''_l the dual of C'_P, it follows immediately from the definition that $m'_P = n''_l$.

4. Returning now to the locus C'_1 (paragraph 1), we have:

$$m'_1 = n_1[(m-1)(n-1)-p].$$

For, each intersection of C'_1 with an arbitrary line l is determined by means of a tangent to C_1 which touches the locus C''_l corresponding to l, and conversely. The number of intersections of C'_1 and l is, therefore,

$$m'_1 = n_1 n''_l = n_1[(m-1)(n-1)-p].$$

5. The locus C'_1 passes through the point of tangency to C of each tangent common to C and C_1; and at each of the remaining intersections with C of these common tangents, C'_1 has a point of inflection, the inflectional tangent coinciding with the line tangent to C at the point of intersection.

6. For, let $x = 0$ be the common tangent, $(0, 0)$ the point of tangency to C, and $(0, 1)$ the point of tangency to C_1, the coordinates being non-homogeneous. Let the equation of the tangent to C_1 in the neighborhood of $x = 0$ be:

$$x + ty - t + \alpha t^2 + \beta t^3 + \dots = 0.$$

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* The following notation will be used: Let m be the order of C; n, its class; δ, the number of its double points; κ, of its cusps; τ, of its double tangents; ι, of its inflections; and p, its genus. To distinguish the corresponding numbers for another curve, as C'_P, subscripts and indices will be affixed; *e. g.*, m'_P is the order of C'_P.

† The locus C'_P was mentioned by Steiner (*Gesamelte Werke*, Vol. II, p. 599, No. 5). He points out that its order equals the class of the dual locus, and states, without proof, that the class of the dual locus is $\frac{1}{2}m(m-1)(2m-3)$ (*ib.*, *id.*, p. 589). This result is correct, provided that C has no multiple points. Sporer, in an article in the *Mathematisch-Naturwissenschaftliche Mitteilungen von Dr. O. Boklen*, Vol. III, pp. 55–58, 1890, attempted to prove Steiner's formula. His proof, however, is incorrect. For example, his proof is based largely on an auxiliary locus of class m^2 which is affirmed to have $m(m-1)$ double tangents, all passing through one fixed point.

Let the equations of C in the neighborhood of the origin be:

$$x = s^2, \qquad y = as + bs^2 + cs^3 + \ldots,$$

where $a \neq 0$. The equation of the tangent to C at the point whose parameter is s is, then:

$$2ys = x(a + 2bs \ldots) + as^2 - cs^4 \ldots.$$

In order that the point of C whose parameter is s lie on the tangent to C_1 whose parameter is t, we must have:

$$s^2 + t(as + bs^2 \ldots) - t + at^2 \ldots = 0.$$

The relation between s and t in the neighborhood of $s = t = 0$ is, therefore:

$$t = t'^2, \qquad s = t' - \frac{a}{2} t'^2 + \frac{a^2 - 4b - 4a}{8} t'^3 \ldots.$$

To the values $\pm t'$ corresponds the same value of t, but different values of s. They therefore determine two points of C lying on the same tangent to C_1. Substituting the two values of s so obtained in the equation of the tangent to C and determining the intersection, we obtain:

$$x = -t + \left(\frac{a^2}{4} + \frac{4c}{a} + b + a\right) t^2 \ldots,$$

$$y = -\frac{2b + a^2}{2} t + \ldots.$$

This is the equation of a branch of C_1' passing through the origin. It should also be noticed, for the proof of a later theorem (see paragraphs 10 and 15), that, since the coefficients of the first power of t do not vanish, the origin is not a cusp of C_1'.

7. In order to show that, at each of the remaining intersections of C with $x = 0$, C_1' has an inflection, let $(0, a')$, where $1 \neq a' \neq 0$, be another intersection of C with $x = 0$. Let the equations of C in the neighborhood of $(0, a')$ be:

$$x = r, \qquad y = a' + b'r + c'r^2 \ldots.$$

From the condition that the point whose parameter is r lie upon the tangent to C_1 whose parameter is $t = t'^2$, we obtain:

$$r = (1 - a')t'^2 + (a'b' - b' - a)t'^4 \ldots.$$

Substituting in the equation of the tangent, and solving as simultaneous with the equation of the tangent at the point whose parameter is s, we obtain:

$$x = \frac{2a'}{a} t' + \left[\frac{4a'}{a^2}(b' - b) - a' - 1\right] t'^2 \ldots,$$

$$y = a' + \frac{2a'}{a} b't' + b'\left[\frac{4a'}{a^2}(b' - b) - a' - 1\right] t'^2 \ldots.$$

The tangent to this branch of C_1' at $(0, a')$ is $y = a' + b'x$, which coincides with the tangent to C at that point. If, in this equation of the tangent, we substitute the above expressions for x and y, the coefficients of the powers of t', of degree less than three, vanish. Hence, $y = a' + b'x$ is an inflectional tangent to C_1'. Since a generic line through $(0, a')$ meets the curve in only one point, $(0, a')$ is not a cusp (see paragraph 10).

8. The tangents to C_1 in the neighborhood of a tangent passing through a node determine, by their intersections with C, a branch of C_1' having a cusp at the node of C. Hence, at each node of C, C_1' has n_1 cusps. For, let $(0, 0)$ be a node of C and $x = 0$ be a tangent to C_1. Let the equations of C in the neighborhood of $(0, 0)$ be:

$$x = s, \qquad y = a s + b s^2 + c s^3 \ldots . ;$$
$$x = r, \qquad y = a' r + b' r^2 + c' r^3 \ldots . .$$

The tangents to the two branches being distinct, $a \neq a'$. Let the equation of the tangent to C_1 be as in paragraph 6. Imposing the condition that the points s and r lie on the tangent t, we obtain for the locus of the points of intersection of corresponding tangents to C:

$$x = \frac{b-b'}{a-a'} t^2 + 2\frac{(a-a')(c-c') - (a-a')(ab-a'b') - \alpha(a-a')(b-b') + (b-b')^2}{(a-a')^2} t^3 \ldots . ,$$
$$y = \frac{a'b - ab'}{a - a'} t^2 \ldots . .$$

Hence C_1' has a cusp at the origin.

9. Through each cusp of C pass n_1 simple branches each touching the cuspidal tangent. For, let the equations of C in the neighborhood of the origin be:

$$x = s^2, \qquad y = a s^3 + b s^4 + \ldots . ,$$

the equation of the tangent to C_1 in the neighborhood of $x = 0$ being as in paragraph 6. Then the locus of the points of intersection of the tangents to C at the points which lie on the same tangent to C_1 is found to be:

$$x = \frac{1}{3} t + \frac{4c - 3a\alpha}{9a} t^2 + \ldots . ,$$
$$y = -\frac{1}{3} b t^2 \ldots . .$$

These are the equations of a branch of C_1' having the origin for a simple point and touching the cuspidal tangent, $y = 0$, at that point.

10. We have already determined (paragraph 8) δn_1 of the cusps of C_1'.

The total number of cusps of C_1' will now be determined. For this purpose, it will be convenient to distinguish several cases, a number of which have been discussed in the four paragraphs immediately preceding.

11. Consider next, therefore, the form of C_1' in the neighborhood of the point of intersection of a cuspidal tangent to C and the tangent at another intersection with C of a tangent to C_1 through the cusp. In addition to the data of paragraph 9, let $x=0$ meet C at $(0, a')$ and let the form of C in the neighborhood of $(0, a')$ be:

$$x = r, \qquad y = a' + b'r + c'r^2 \ldots\ldots$$

We may suppose, it being projectively no restriction, that $b' \neq 0$. The equations of C_1' are then of the form:

$$x = -\frac{a'}{b'} - \frac{3\,a\,a'}{2\,b'^2}\,t + \ldots\ldots,$$
$$y = -\frac{3\,a\,a'}{b'}\,t + \ldots\ldots$$

Hence C_1' does not have a cusp at the point of intersection of the cuspidal tangent with the tangent at $(0, a')$.

12. Consider, next, two branches of C whose equations are:

$$x = s, \qquad y = a + bs + cs^2 + \ldots\ldots;$$
$$x = r, \qquad y = a' + b's + c's^2 + \ldots\ldots,$$

where $a = a'$ and $b \neq b'$, the equation of the tangent to C_1 being as in paragraph 6. It will now be determined under what conditions the tangents at $(0, a)$ and $(0, a')$ determine a cusp of C_1'.

13. The locus of the point of intersection of corresponding tangents is:

$$x = -\frac{a-a'}{b-b'} + 2\left[c(1-a) - c'(1-a')\right]\frac{a-a'}{(b-b')^2}\,t + \ldots\ldots,$$
$$y = \frac{a'b - ab'}{b-b'} + 2\left[cb'(1-a) - bc'(1-a')\right]\frac{a-a'}{(b-b')^2}\,t + \ldots\ldots$$

If this branch of C_1' has a cusp at $t=0$, the coefficients of the first power of t must vanish. But $a \neq a'$ and $b \neq b'$; hence the conditions for a cusp reduce to

$$c(1-a) = 0, \qquad c'(1-a') = 0.$$

But $c=0$ or $c'=0$ is the condition that one of the two branches of C have a point of inflection on $x=0$, and $a=1$ or $a'=1$ is the condition that one of the two branches pass through the point of tangency $(0, 1)$ of $x=0$ to C_1. Since C and C_1 are in general position with respect to each other, the above two equations can not be satisfied simultaneously.

14. There still remains to be considered the case in which $x=0$ is an inflectional tangent to C_1. Let the equation of the tangent to C_1, in the neighborhood of the inflectional tangent $x=0$, be:

$$x+t^2y-t^2+\beta t^3+\ldots\ldots=0.$$

Let the equations of C, in the neighborhood of two of its intersections with $x=0$, be:

$$x=s, \qquad y=a+cs^2+\ldots\ldots;$$
$$x=r, \qquad y=r+c'r^2+\ldots\ldots,$$

where $0 \neq a \neq 1$. Then the locus of the point of intersection of corresponding tangents is:

$$x=a+2a(c+c'-ac)t^2-2\beta a(c-c')t^3\ldots\ldots,$$
$$y=a+2ac(1-a)t^2-2\beta act^3\ldots\ldots.$$

Hence, C_1' has a cusp at the point of intersection (a, a) of the tangents to C at $(0, 0)$ and $(0, a)$. Hence, each point of intersection of the tangents to C at the m intersections with C of each inflectional tangent to C_1 is a cusp of C_1'.

15. Hence, in general, the number of cusps of C_1' is:

$$x_1'=n_1\delta+\tfrac{1}{2}m(m-1)\iota_1.$$

16. The genus, p_1', of C_1' will next be determined. For this purpose, the following auxiliary locus will be used: Let q be an arbitrary line in the plane, and Q an arbitrary point on it. Draw the n_1 tangents from Q to C_1. These n_1 tangents to C_1 determine $\frac{1}{2}n_1m(m-1)$ points of C_1'. Draw the lines joining these $\frac{1}{2}n_1m(m-1)$ points to Q. Then the auxiliary locus in question is the locus of these joining lines as Q describes q. This line locus is in $(1, 1)$ correspondence with C_1'. Hence its genus is equal to that of C_1'.

17. The line q is an m_1'-fold line of this locus. For, each intersection of C_1' with q determines a line of the locus coincident with q. Through an arbitrary point of q pass $\frac{1}{2}n_1m(m-1)$ other lines of the locus. Hence the class of the envelope of this system of lines is $m_1'+\frac{1}{2}n_1m(m-1)$.

18. At each intersection, Q, other than the points of tangency, of q with the envelope of these lines or with an inflectional tangent of the envelope, two of the $\frac{1}{2}n_1m(m-1)$ lines through Q become consecutive, and conversely. But if two of these lines are consecutive, the corresponding points of C_1' must be consecutive. This may happen in either of two ways. First, points determined by different tangents to C_1 from Q may become consecutive. The two tangents to C_1 which determine them must then be consecutive. There are $m_1+\iota_1$ points Q at which two of the tangents to C_1 become consecutive. At each of

these points $\frac{1}{2}m(m-1)$ pairs of the corresponding points of C_1' become consecutive. Second, two points determined by the same tangent to C_1 may become consecutive. When this happens, the tangents to C at two intersections with C of this tangent to C_1 must be consecutive; that is, such a tangent to C_1 must either touch C or pass through a cusp. The number of such tangents to C_1 is $(n+\varkappa)n_1$. Of the $\frac{1}{2}m(m-1)$ points of C_1' determined by each of them, $m-2$ pairs are consecutive. Hence the sum of the order and of the number of inflectional tangents of the envelope of the line locus determined by q is:

$$2m_1' + \tfrac{1}{2}m(m-1)(m_1+\iota_1) + (n+\varkappa)(n-2)n_1 .$$

19. It now follows, by Plucker's equations, that the genus of this locus, that is, the genus p_1' of C_1', is:

$$p_1' = \tfrac{1}{4}m(m-1)(m_1+\iota_1) + \tfrac{1}{2}(n+\varkappa)(m-2)n_1 - \tfrac{1}{2}m(m-1)n_1 + 1.$$

The three quantities m_1', $\varkappa_1'$ and p_1' having been determined, the rest of Plucker's numbers follow by means of Plucker's equations.

20. By considerations dual to those employed in the case of the locus C_1', the corresponding properties of the following locus can be obtained: In the plane of an arbitrary algebraic curve C, let there be given a second arbitrary algebraic curve C_1. From a point Q of C_1 draw the n tangents to C and construct the lines joining their points of tangency. Then the locus in question is the locus of these joining lines as Q describes C_1. The envelope of these lines will be referred to as C_1''.

21. Since the loci C_P' and C_l'' (paragraphs 2 and 3) are particular cases of C_1' and C_1'', from the discussion of the latter we obtain at once corresponding properties of the former. The curves C_P', corresponding to the points of the plane, form a doubly infinite system. The number of curves of this system which touch a given line l when P describes a second given line $\bar{l}$ is readily determined. In fact, if C_P touches l, then P lies upon C_l''. The number required is therefore the number of intersections of C_l'' and $\bar{l}$; *i. e.*, m_l''. More generally, if P describes a given curve $\bar{C}_1$, the number of the corresponding curves which touch a second given curve C_1 is $\bar{m}_1 m_1''$.

22. The envelope of the curves C_P', as P describes a given curve C_1, may also be determined. Let Q be a point in the plane. The intersections with C_1 of the lines joining the points of tangency of the tangents to C from Q determine $\frac{1}{2}m(m-1)m_1$ points P on C_1 whose C_P' curves pass through Q. In order that two of these curves be consecutive, the corresponding points P must be consecutive. Hence, either two of the joining lines are consecutive or

else one of the joining lines touches C_1. In the first case Q lies either on C or on an inflectional tangent to C. For such a point Q, $n-2$ of the joining lines are consecutive. In the second case Q lies upon C_1'. Hence, the envelope required is composed of C and its inflectional tangents, each counted $(n-2)\,m_1$ times together with C_1'.

23. Of especial interest are the particular cases of C_1' and C_1'' in which C_1 coincides with C. The particular case of C_1' may be defined as follows: Let t be any tangent to C. At the $m-2$ intersections, other than the point of tangency, of t with C construct the tangents to C. Then the locus in question is the locus of the $\frac{1}{2}(m-2)(m-3)$ intersections of these $m-2$ tangents as t describes the system of tangents to C. This locus will be referred to as the locus C'.

24. The foregoing discussion of the locus C_1' is applicable only in part to C'. To determine the order of C' we first determine, as for C_1', the number of tangents to C which touch an arbitrary envelope C_l''. From this number, $n[(m-1)(n-1)-p]$, we must, however, subtract the number $4\tau+3\iota$ of these tangents which coincide with the double and inflectional tangents of C, and the number $m+2m(n-2)$ of them which pass through the points of intersection of C and l. We have, therefore, for the order of C':

$$m' = n[(m-1)(n-1)-p] - 4\tau - 3\iota - m(2n-3).$$

25. From each node of C, $n-4$ tangents to C can be drawn. The tangents to C in the neighborhood of such a tangent determine, by their intersections with C in the neighborhood of the node, a branch of C' having a cusp at the node. It is similarly seen that, through each cusp of C, pass $n-3$ branches of C', each touching the cuspidal tangent. The locus C' passes through both points of tangency of each double tangent and, at each of the remaining intersections of these tangents with C, has two inflections such that both inflectional tangents coincide with the tangent to C at that point. The $\frac{1}{2}(m-3)(m-4)$ points of intersection of the tangents to C at the points of intersection with C, other than the points of inflection, of each inflectional tangent, are cusps of C'.

26. At each point of intersection with C, distinct from the cusp, of each cuspidal tangent, C' touches C. For, let the form of C in the neighborhood of the cusp be:

$$x = t^3, \qquad y = a\,t^2 + b\,t^3 + c\,t^4 + \ldots,$$

and in the neighborhood of another intersection with the cuspidal tangent, be:

$$x = s, \qquad y = a' + b's + c's^2 \ldots.$$

The tangent at the point whose parameter is t meets C again in the neighborhood of the cusp and also in the neighborhood of $(0, a')$. The locus of the point of intersection of the tangents at these two points is a branch of C' whose equations are:

$$x = -\frac{3a'}{4a}t + \frac{9a'(b'-b)}{16a^2}t^2 + \ldots,$$

$$y = a' - \frac{3a'}{4a}b't + \ldots.$$

Hence C' touches C at $(0, a')$. It should also be noticed that this point is not a cusp of C'.

27. At each intersection with C, distinct from the inflection, of each inflectional tangent, C' has a ramphoid cusp. For, let the form of C in the neighborhood of the point of inflection be:

$$x = t^3, \qquad y = at + bt^2 + ct^3 + \ldots,$$

and in the neighborhood of $(0, a')$, be:

$$x = s, \qquad y = a' + b's^2 + \ldots.$$

The tangent at the point whose parameter is t determines by its intersections with C in the neighborhood of $(0, 0)$ and $(0, a')$ a branch of C' whose equations are:

$$x = \frac{12a'}{a^2}t^2 + \frac{12ba' - 16a^2}{a^2}t^3 + \ldots,$$

$$y = a' + \frac{63a'^2b'}{a^2}t^4 + \ldots.$$

Hence the approximate form of C' in the neighborhood of $(0, a')$ is

$$y - a' - Mx^2 = Nx^{\frac{5}{2}},$$

where M and N are constants. Hence C' has a ramphoid cusp at $(0, a')$, the cuspidal tangent being the tangent $y = a'$ to C at that point.

28. We have, therefore, in general, for the number of cusps of C', the expression

$$\varkappa' = \delta(n-4) + \tfrac{1}{2}\iota(m-2)(m-3).$$

29. The genus of C' may be determined in a manner analogous to that by which C_1' was obtained (paragraph 16). On an arbitrary line q take a point Q Draw the n tangents from Q to C, thus determining $\frac{1}{2}n(m-2)(m-3)$ points of C'. Join these points to Q. As Q describes q, these joining lines envelope a curve of class $m' + \frac{1}{2}n(m-2)(m-3)$. The sum of the order and of the number of inflections of the envelope is

$$2m' + \tfrac{1}{2}(m+\iota)(m-2)(m-3) + [2\tau + \varkappa(n-3)](m-4).$$

Hence the genus of this envelope, that is, the genus of C', is:

$$p' = \tfrac{1}{4}(m + \iota - 2n)(m - 2)(m - 3) + \tfrac{1}{2}[2\tau + \varkappa(n - 3)](m - 4) + 1.$$

30. With the assistance of the locus C' or of its dual, it is possible to determine the number of tangents t to an arbitrary algebraic curve C such that the tangents to C at two of the intersections of t with C meet at a point T which lies on C.* Every such point T is evidently a point of intersection of C

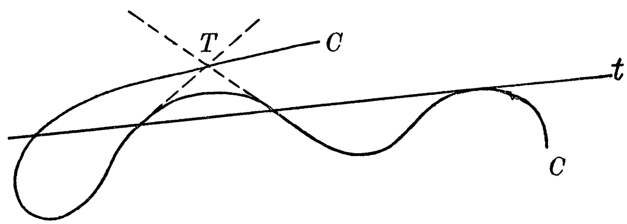

FIG. 2.

and C', and every such line t is tangent both to C and the dual of C'. The number of points of intersection of C and C' is:

$$m\{n[(m-1)(n-1)-p] - 4\tau - 3\iota - m(2n-3)\}.$$

From this number should be subtracted the number of intersections due to the singularities of C (paragraphs 25–27). The resulting number of points T or of tangents t is:

$$\begin{aligned} &mn[(m-1)(n-1)-p] - m^2(2n-3) - 8m\tau - 7m\iota \\ &\qquad - 4n\delta + 16\delta - 3n\varkappa + 15\varkappa + 14\tau - 2m\varkappa + 12\iota. \end{aligned}$$

31. The number in question can also be obtained by subtracting from the total number of common tangents to C and the dual of C' the number of common tangents due to the singularities of C. We are thus led to the expression:

$$\begin{aligned} &mn[(m-1)(n-1)-p] - n^2(2m-3) - 8\delta n - 7n\varkappa \\ &\qquad - 4m\tau + 16\tau - 3m\iota + 15\iota + 14\delta - 2n\iota + 12\varkappa. \end{aligned}$$

This expression is equivalent to the preceding one.

* On certain algebraic curves, every point has the property in question. In this case C itself is a component of C' and of its dual. Weil has shown, in the *Nouvelles Annales de Mathématiques*, Ser. 3, Vol. XII, pp. 93–95, that, in the case of the curves $y^m = x^p$, if any tangent, t, to the curve be drawn and, at the points of intersection of t with the curve, the tangents to the curve be constructed, then all the points of intersection of these tangents lie on the given curve.